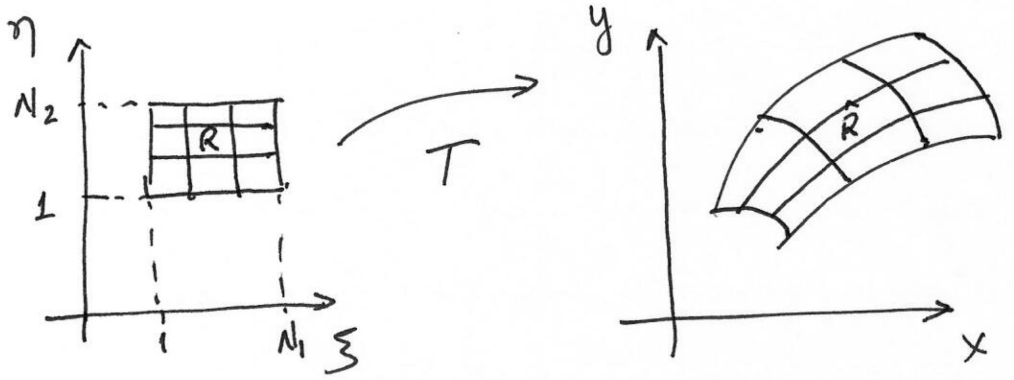
Grid Quality

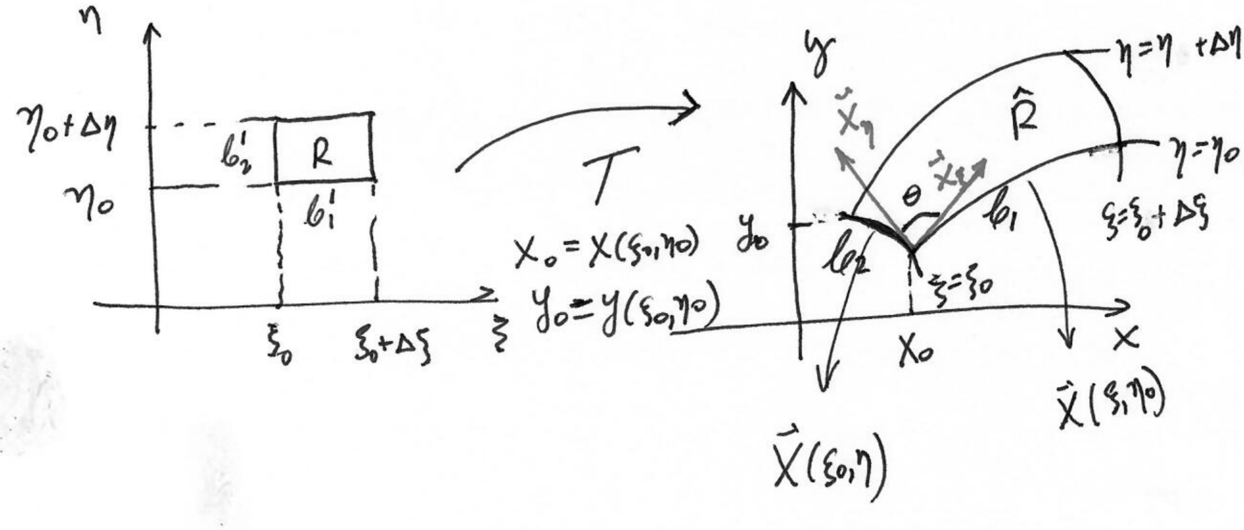
Consider

$$T: D' \longrightarrow D$$

$$(\xi,\eta) \longrightarrow (x,y) = (x(\xi,\eta), y(\xi,\eta)) = \vec{X}(\xi,\eta)$$



Now, consider the cell R and $\hat{R}$ (its image).



b_1: $\vec{X}(\xi,\eta_0) \Rightarrow \vec{X}_\xi(\xi_0,\eta_0)$: tang. vector at (x_0,y_0) to b_1.

b_2: $\vec{X}(\xi_0,\eta) \Rightarrow \vec{X}_\eta(\xi_0,\eta_0)$: tang. vector at (x_0,y_0) to b_2.

$$\vec{X}_\xi(\xi,\eta) = \left(x_\xi(\xi,\eta),\ y_\xi(\xi,\eta)\right)$$

$$\vec{X}_\eta(\xi,\eta) = \left(x_\eta(\xi,\eta),\ y_\eta(\xi,\eta)\right)$$

Thm.- For the transformation T described above the following properties are satisfied.

i) $\cos\theta = \dfrac{\beta}{(\alpha\gamma)^{1/2}}$, at (x_0, y_0),

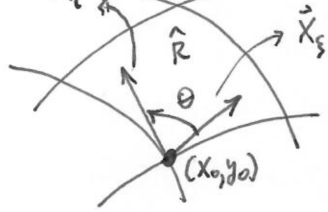
where $\alpha = x_\eta^2 + y_\eta^2$, $\beta = x_\xi x_\eta + y_\xi y_\eta$, $\gamma = x_\xi^2 + y_\xi^2$.

ii) Area of the curvilinear cell $\hat{R}$, image of the rectangular cell R in the computational plane, can be approximated by

$$\text{Area of } \hat{R} \approx \left|\left(x_\xi y_\eta - x_\eta y_\xi\right)\right| \Delta\xi\,\Delta\eta$$
$$= \left|\text{Jacobian of } T\right| \Delta\xi\,\Delta\eta = |J|\Delta\xi\,\Delta\eta.$$

Proof.-

i)



$$\langle \vec{X}_\xi, \vec{X}_\eta \rangle = \|\vec{X}_\xi\|\,\|\vec{X}_\eta\| \cos\theta$$
$$= \left(x_\xi^2 + y_\xi^2\right)^{1/2} \left(x_\eta^2 + y_\eta^2\right)^{1/2} \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\langle \vec{X}_\xi, \vec{X}_\eta \rangle}{\left(x_\xi^2 + y_\xi^2\right)^{1/2}\left(x_\eta^2 + y_\eta^2\right)^{1/2}} = \frac{x_\xi x_\eta + y_\xi y_\eta}{\left(x_\xi^2 + y_\xi^2\right)^{1/2}\left(x_\eta^2 + y_\eta^2\right)^{1/2}}.$$

Defining,

$$\mathcal{H} = x_\xi^2 + y_\xi^2, \quad \beta = x_\xi x_\eta + y_\xi y_\eta, \quad \alpha = x_\eta^2 + y_\eta^2$$

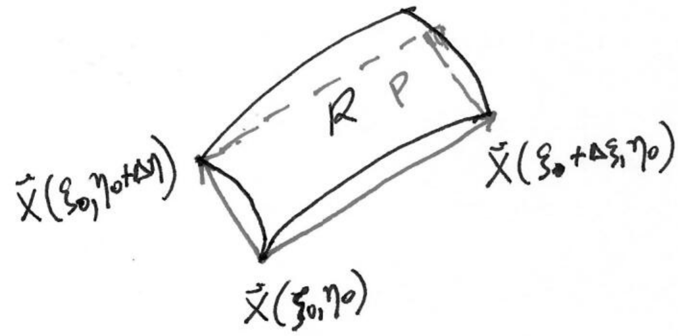
we obtain

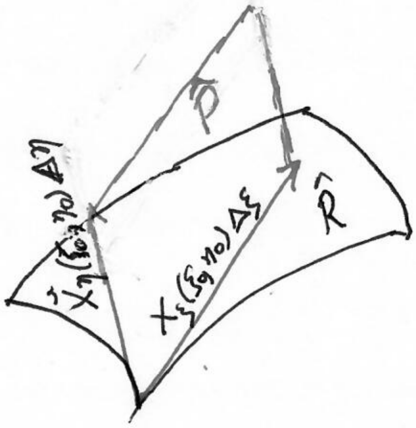
$$\cos\theta = \frac{\beta}{(\mathcal{H}\alpha)^{1/2}} \quad \text{at } (x_0, y_0) \qquad (24.1)$$

If $\theta = 90°$ then we will say that the grid lines are orthogonal at (x_0, y_0).

ii) Proof of

$$\text{Area of } \hat{R} \simeq \left\| \begin{matrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{matrix} \right\| \Delta\xi\, \Delta\eta = |J|\, \Delta\xi\, \Delta\eta$$





So area of $R \simeq$ area of parallelogram P

formed by $\vec{x}(\xi_0+\Delta\xi, \eta_0) - \vec{X}(\xi_0, \eta_0)$

and $\vec{X}(\xi_0, \eta_0+\Delta\eta) - \vec{X}(\xi_0, \eta_0)$.

Which in turn can be approximated by the area of $\hat{P}$

formed by

$$\vec{X}(\xi_0+\Delta\xi, \eta_0) - \vec{X}(\xi_0, \eta_0) \simeq \vec{X}_\xi(\xi_0, \eta_0)\,\Delta\xi$$

$$\vec{X}(\xi_0, \eta_0+\Delta\eta) - \vec{X}(\xi_0, \eta_0) \simeq \vec{X}_\eta(\xi_0, \eta_0)\,\Delta\eta$$

Now,

$$\text{area of } \hat{P} = \left| \left(\vec{X}_\xi (\xi_0, \eta_0) \Delta\xi \right) \times \left(\vec{X}_\eta (\xi_0, \eta_0) \Delta\eta \right) \right|$$

$$= \left| \left(\vec{X}_\xi \times \vec{X}_\eta \right) (\xi_0, \eta_0) \right| \Delta\xi \, \Delta\eta .$$

$$= \text{norm} \left(\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_\xi \Delta\xi & y_\xi \Delta\xi & 0 \\ x_\eta \Delta\eta & y_\eta \Delta\eta & 0 \end{vmatrix} \right)$$

$$= \text{norm} \left[\left(x_\xi y_\eta - x_\eta y_\xi \right)^{(\xi_0, \eta_0)} \Delta\xi \Delta\eta \, \hat{k} \right] = \left| \left(x_\xi y_\eta - x_\eta y_\xi \right) (\xi_0, \eta_0) \right| \Delta\xi \Delta\eta$$

$$= J (\xi_0, \eta_0) \Delta\xi \Delta\eta .$$

and from the previous discussion

Area of Curvilinear cell $\hat{R} \simeq$ area of $\hat{P}$

when N_1 and N_2 are large

or

Area of Curvilinear cell $\hat{R} \simeq \left| J (\xi_0, \eta_0) \right| \Delta\xi \Delta\eta .$

For a grid defined by the points

$$(x_{i,j}, y_{i,j}), \qquad \begin{matrix} 1 \le i \le N_1 \\ 1 \le j \le N_2 \end{matrix}$$

where $x_{i,j} = x(\xi_i, \eta_j)$, $\quad y_{i,j} = y(\xi_i, \eta_j)$

the local angle at $(x_{i,j}, y_{i,j})$ is given by

$$\theta_{i,j} = \arccos\left(\frac{\beta_{ij}}{(\gamma_{i,j}\, \alpha_{i,j})^{1/2}} \right)$$

To compute $\beta_{i,j}$, $\gamma_{i,j}$ and $\alpha_{i,j}$ finite difference approximations of the derivatives can be employed.

For example,

$$\alpha_{i,j} = (x_\eta)^2_{i,j} + (y_\eta)^2_{i,j} \cong \left(\frac{x_{i,j+1} - x_{i,j-1}}{2\Delta\eta} \right)^2 + \left(\frac{y_{i,j+1} - y_{i,j-1}}{2\Delta\eta} \right)^2$$

Similarly,

$$\beta_{i,j} = (x_\xi)_{ij}(x_\eta)_{ij} + (y_\xi)_{ij}(y_\eta)_{ij}$$

$$= \left(\frac{x_{i+1,j} - x_{i-1,j}}{2\Delta\xi}\right)\left(\frac{x_{i,j+1} - x_{i,j-1}}{2\Delta\eta}\right)$$

$$+ \left(\frac{y_{i+1,j} - y_{i-1,j}}{2\Delta\xi}\right)\left(\frac{y_{i,j+1} - y_{i,j-1}}{2\Delta\eta}\right) =$$

and

$$\gamma_{i,j} = \left(\frac{x_{i+1,j} - x_{i-1,j}}{2\Delta\xi}\right)^2 + \left(\frac{y_{i+1,j} - y_{i-1,j}}{2\Delta\xi}\right)^2$$

Therefore,

$$\theta_{ij} = \arccos(A_{ij})$$

where

$$A_{ij} = \frac{(x_{i+1,j} - x_{i-1,j})(x_{i,j+1} - x_{i,j-1}) + (y_{i+1,j} - y_{i-1,j})(y_{i,j+1} - y_{i,j-1})}{\left[\left((x_{i,j+1} - x_{i,j-1})^2 + (y_{i,j+1} - y_{i,j-1})^2\right)\left((x_{i+1,j} - x_{i-1,j})^2 + (y_{i+1,j} - y_{i-1,j})^2\right)\right]^{1/2}}$$

Also, Area of $\hat{R}_{ij} = |J_{ij}|\,\Delta\xi\,\Delta\eta = \left((x_\xi)_{ij}(y_\eta)_{ij} - (x_\eta)_{ij}(y_\xi)_{ij}\right)\Delta\xi\,\Delta\eta$

$$= \frac{1}{4}\left[(x_{i+1,j} - x_{i-1,j})(y_{i,j+1} - y_{i,j-1}) - (x_{i,j+1} - x_{i,j-1})(y_{i+1,j} - y_{i-1,j})\right]$$

Also, the Jacobian at this point is given by

$$J_{i,j} \simeq \left(x_\xi y_\eta - x_\eta y_\xi \right)_{i,j} \Delta\xi \Delta\eta \qquad (25.1)$$

In both formulas (24.1) and (25.1), the approximated values can be obtained by using centered differences for the derivatives involved.

Another important parameters used to measure the orthogonality of the grid are

i) $$MDO = \max_{\substack{2 \le i \le N_1 - 1 \\ 2 \le j \le N_2 - 1}} |90^\circ - \theta_{ij}|$$

ii) $$ADO = \left(\frac{1}{N_1 - 2}\right)\left(\frac{1}{N_2 - 2}\right) \sum_{i=2}^{N_2 - 1} \sum_{j=2}^{N_1 - 1} \left(|90^\circ - \theta_{i,j}| \right)$$

The first one (i) measure the maximum deviation from orthogonality. And (ii) is a global measure of the grid ortogonality (Average deviation from orthogonality).

Rmk: In the above definitions of ADO and MDO, we are only including interior grid points.